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SOLVING IRRATIONAL EQUATIONS AND INEQUATIONS THROUGH TRIANGLE BISECTOR PROPERTIES

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Rezumat. Pentru ecuațiile algebrice de gradul întâi, doi, trei și patru există metode generale de rezolvare a lor și aceste metode sunt cunoscute destul de bine. De asemenea este demonstrat că ecuațiile algebrice de gradul cinci și mai mare, în caz general, nu pot fi rezolvate în radicali. Referitor la ecuațiile iraționale situația este cu totul alta. Chiar rezolvând ecuații iraționale ce conțin radicali de ordinul doi de acum se întâlnesc anumite dificultăți. Evident că situația devine și mai dificilă atunci când ecuația irațională conține radicali de ordin diferit. Astfel, în acest articol, se propune o metodă pur geometrică de rezolvare a ecuațiilor și inecuațiilor iraționale ce conțin radicali de gradul doi.
Cuvinte cheie: ecuație irațională, bisectoare, soluție, metodă, inecuație irațională.

Abstract. For algebraic equations of the first, second, third, and fourth degrees there are general methods of solving them, and these methods are fairly well known. It is also shown that algebraic equations of degree five and higher, in general, cannot be solved in radicals. With regard to irrational equations, the situation is completely different. Even solving irrational equations containing radicals of the second order now encounter certain difficulties. Obviously, the situation becomes even more difficult when the irrational equation contains radicals of different order. Thus, in this article, a purely geometric method for solving irrational equations and inequations containing radicals of the second degree is proposed.
Keywords: irrational equation, bisector, solution, method, irrational inequation.

I. Introduction

In the process of solving irrational equations we encounter different "obstacles", because there is no universal method that would allow us to solve the given equation and at the same time be a rational and "convenient" method for us [1, 2, 3].

In this article we will indicate geometric methods for solving irrational equations, which in certain cases are quite "convenient" for us and at the same time are the simplest.

II. The case of the interior bisector of the triangle

First we will prove the following lemma:

Lemma 1. *Let* $ab > 0$ *and* $c > 0$, *where* a, b, c *are real numbers. The number* x_0 *is a solution of the equation* $\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x^2 - 2bdx} = c$, *if and only if the number* $-x_0$ *is a solution of the equation* $\sqrt{a^2 + x^2 + 2adx} + \sqrt{b^2 + x^2 + 2bdx} = c$.

Demonstration. We will have

$$\sqrt{a^2 + x_0^2 - 2adx_0} + \sqrt{b^2 + x_0^2 - 2bdx_0} = c \Leftrightarrow$$

$$\Leftrightarrow \sqrt{a^2 + (-x_0)^2 + 2ad \cdot (-x_0)} + \sqrt{b^2 + (-x_0)^2 + 2bd \cdot (-x_0)} = c.$$

The Lemma 1 is proved.

We next prove the following theorem:

Theorem 1. *Let* $ab > 0$, $c > 0$ *and* $0 < d < 1$, where a, b, c, d *are real numbers*. *Irrational equation of the form*

$$\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x^2 - 2bdx} = c, \quad (1)$$

which satisfies the condition

$$c^2 = (a + b)^2 - 4abd^2 \quad (2)$$

admits a unique solution.

Demonstration. We indicate the following two methods.

Method 1. Algebraic method. We write the irrational equation (1) in the form:

$$\sqrt{a^2 + x^2 - 2adx} = c - \sqrt{b^2 + x^2 - 2bdx}.$$

By raising both sides to the second power

$$2c\sqrt{b^2 + x^2 - 2bdx} = (b^2 + c^2 - a^2) + 2d(a - b)x,$$

where from

$$4(d^2 (a - b)^2 - c^2)x^2 + (8bdc^2 + 4d(b^2 + c^2 - a^2)(a - b))x +$$
$$+ (b^2 + c^2 - a^2)^2 - 4b^2c^2 = 0. \quad (3)$$

According to condition (2)

$$c^2 = (a + b)^2 - 4abd^2.$$

So

$$4(d^2(a - b)^2 - c^2) = 4(a + b)^2(d^2 - 1),$$
$$8bdc^2 + 4d(b^2 + c^2 - a^2)(a - b) = - 16abd(a + b)(d^2 - 1),$$
$$(b^2 + c^2 - a^2)^2 - 4b^2c^2 = 16a^2b^2d^2(d^2 - 1).$$

Therefore, equation (3) will take the form:

$$4(a + b)^2x^2 - 16abd(a + b)x + 16a^2b^2d^2 =0,$$

where from

$$(2(a + b)x - 4abd)^2 = 0 \text{ or } x = \frac{2abd}{a+b}.$$

Method 2. Geometric method. Two cases are possible.

Case 1. $a > 0$, $b > 0$ and $a \geq b$. Since $0 < d < 1$, then and $0 < d^2 < 1$. After some elementary transformations we obtain $a - b < c < a + b$, that is, a, b and c are the lengths of the sides of a triangle. Therefore, the irrational equation (1) admits only positive solutions. We construct a triangle ABC with sides $AB = c$, $BC = a$ and $AC = b$ (fig. 1).

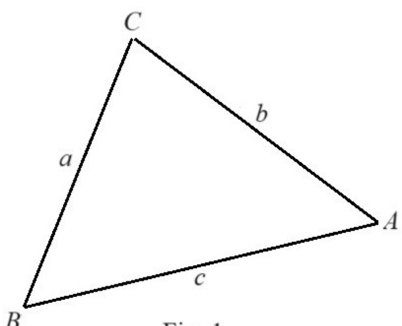

Fig. 1

Let $m(\angle ACB) = \gamma$. According to the cosine theorem

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

But according to condition (2)

$$c^2 = (a + b)^2 - 4abd^2.$$

Thus

$$a^2 + b^2 - 2ab\cos\gamma = (a + b)^2 - 4abd^2,$$

where from $\cos\gamma = 2d - 1$. Because $\cos\gamma = 2\cos^2\frac{\gamma}{2} - 1$, it follows that $d = \cos\frac{\gamma}{2}$. Then the irrational equation (1) can be written in the form:

$$\sqrt{a^2 + x^2 - 2ax\cos\frac{\gamma}{2}} + \sqrt{b^2 + x^2 - 2bx\cos\frac{\gamma}{2}} = c. \quad (4)$$

We take the bisector of the angle ACB. Let x be one of the solutions of equation (1). We place on this bisector the segment $CD = x$ (fig. 2) and prove that point D belongs to side $[AB]$.

According to the theorem of cosines, for the triangle DCB we will obtain:

$$BD = \sqrt{a^2 + x^2 - 2ax\cos\frac{\gamma}{2}}.$$

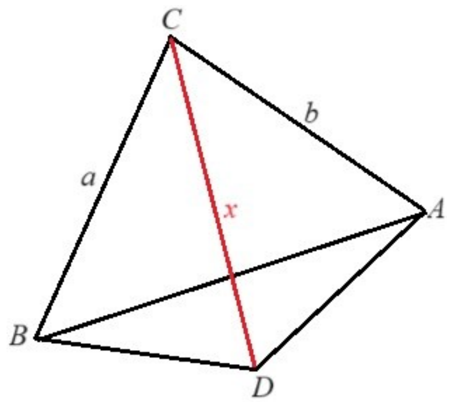

Fig. 2

Analogously, for the triangle ACD we will obtain

$$DA = \sqrt{b^2 + x^2 - 2bx\cos\frac{\gamma}{2}}.$$

Then

$$BD + DA = \sqrt{a^2 + x^2 - 2ax\cos\frac{\gamma}{2}} + \sqrt{b^2 + x^2 - 2bx\cos\frac{\gamma}{2}} = c = AB$$

and therefore $D \in [AC]$ (fig. 3). Therefore, x is a unique solution, because the bisector CD of the angle ACB is unique. Then, according to the planimetry theorem, the length of the bisector CD can be calculated according to the formula:

$$x = \frac{2ab\cos\frac{\gamma}{2}}{a + b}.$$

<u>Case 2.</u> $a < 0$ and $b < 0$. In this case, we first solve the irrational equation of the form:

$$\sqrt{a^2 + x^2 - 2|x|\cos\frac{\gamma}{2}} + + \sqrt{b^2 + x^2 - 2|b|x\cos\frac{\gamma}{2}} = c. \quad (5)$$

Let x_0 be a solution of the irrational equation (1). According to Lemma 1, equation (5) admits a unique solution $-x_0$, because equation (1) admits a unique positive solution. Theorem 1 is proved.

Corollary 1. *Let* $ab < 0$, $c > 0$ *and* $0 < d < 1$. *The irrational equation of the form*:

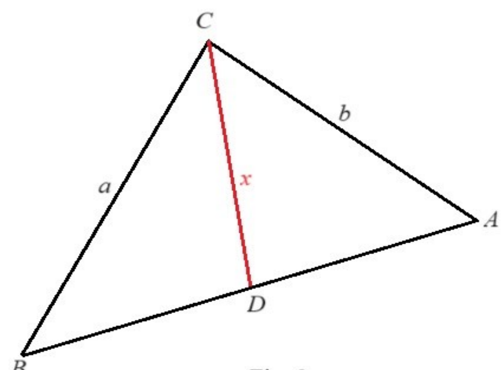

Fig. 3

$$\sqrt{a^2 + x^2 + 2adx} + \sqrt{b^2 + x^2 - 2bdx} = c, \tag{6}$$

which satisfies the condition

$$c^2 = (-a + b)^2 + 4abd^2 \tag{7}$$

admits a unique solution.

Corollary 2. *Let* $ab > 0$, $0 < d < 1$ *and* $c_0^2 = (a + b)^2 - 4abd^2$. *Then*:

1. *If* $c < c_0$, *then equation* (1) *does not admit any solution*.
2. *If* $c = c_0$, *then equation* (1) *admits a unique solution*.
3. *If* $c > c_0$, *then equation* (1) *admits two distinct solutions*.

Corollary 3. *Let* $ab \neq 0$, $d = 0$ *and* $c_0 = |a| + |b|$. *Then*:

1. *If* $c < c_0$, *then equation* (1) *does not admit any solution*.
2. *If* $c = c_0$, *then equation* (1) *admits a unique solution*.
3. *If* $c > c_0$, *then equation* (1) *admits two distinct solutions*.

III. Conclusions for the case of the interior bisector of the triangle

The results presented in this article lead us to an interesting method for solving irrational inequations, namely:

Let $ab > 0$, $c > 0$, $0 < d < 1$ and $c^2 = (a + b)^2 - 4abd^2$. Then the irrational inequation of the form:

1. $\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x^2 - 2bdx} < c$ does not admit any solution.
2. $\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x^2 - 2bdx} > c$ admit the solutions
$$x \in \left(-\infty; \frac{2abd}{a+b}\right) \cup \left(\frac{2abd}{a+b}; +\infty\right).$$
3. $\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x_2 - 2bdx} \leq c$ admit the solution $x = \frac{2abd}{a+b}$.
4. $\sqrt{a^2 + x^2 - 2adx} + \sqrt{b^2 + x_2 - 2bdx} \geq c$ admit the solutions $x \in R$.

In order to solve the irrational inequalities that have the forms indicated above, we must, first, solve the irrational equation of the respective form, then apply the ones mentioned above in this article.

IV. The case of the exterior bisector of the triangle

For the case of the exterior bisector of the triangle the form of the irrational equation has the form:

$$\sqrt{a^2 + x^2 + 2adx} - \sqrt{b^2 + x^2 - 2bdx} = c, \tag{8}$$

where $ab > 0$, $c > 0$, $0 < d < 1$ with the condition (fig. 4)

$$c^2 = (a - b)^2 + 4abd^2. \tag{9}$$

The analogue of Lemma 1 for the exterior bisector is

Lemma 2. *Let* $ab > 0$ *and* $c > 0$, where a, b, c *are real numbers. The number* x_0 *is a solution of the equation*

$$\sqrt{a^2 + x^2 + 2adx} - \sqrt{b^2 + x^2 - 2bdx} = c,$$

if and only if the number $-x_0$ *is a solution of the equation* $\sqrt{a^2 + x^2 - 2adx} - \sqrt{b^2 + x^2 + 2bdx} = c$.

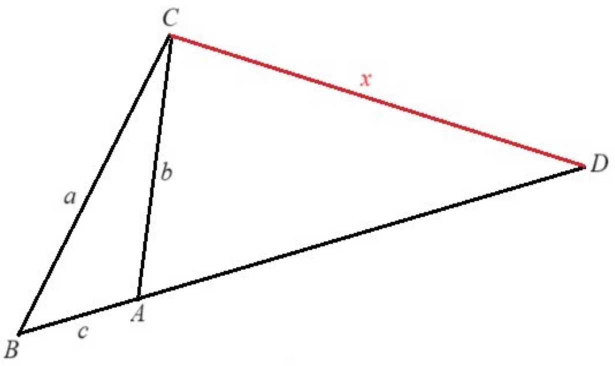

Fig. 4

The analogue of Theorem 1 for the exterior bisector is

Theorem 2. *Let* $ab > 0$, $c > 0$ *and* $0 < d < 1$, where a, b, c, d *are real numbers. Irrational equation of the form*

$$\sqrt{a^2 + x^2 + 2adx} - \sqrt{b^2 + x^2 - 2bdx} = c, \tag{10}$$

which satisfies the condition

$$c^2 = (a - b)^2 + 4abd^2 \tag{11}$$

admits a unique solution.

From Theorem 2 we obtain:

Corollary 4. *Let* $ab < 0$, $c > 0$ *and* $0 < d < 1$. *The irrational equation of the form*:

$$\sqrt{a^2 + x^2 - 2adx} - \sqrt{b^2 + x^2 - 2bdx} = c, \tag{12}$$

which satisfies the condition

$$c^2 = (a + b)^2 - 4abd^2 \tag{13}$$

admits a unique solution.

Corollary 5. *Let* $ab > 0$, $0 < d < 1$ *and* $c_0^2 = (a + b)^2 + 4abd^2$. *Then*:

1. *If* $c < c_0$, *then equation* (8) *admits two distinct solutions*.
2. *If* $c = c_0$, *then equation* (8) *admits a unique solution*.
3. *If* $c > c_0$, *then equation* (8) *does not admit any solution*.

Corollary 6. *Let* $ab \neq 0$, $d = 0$ *and* $c_0 = \big||a| - |b|\big|$. *Then*:

1. *If* $c < c_0$, *then equation* (8) *admits two distinct solutions*.
2. *If* $c = c_0$, *then equation* (8) *admits a unique solution*.
3. *If* $c > c_0$, *then equation* (8) *does not admit any solution*.

V. Conclusions for the case of the exterior bisector of the triangle

Let $ab > 0$, $c > 0$, $0 < d < 1$ and $c^2 = (a - b)^2 + 4abd^2$. Then the irrational inequation of the form:

1. $\sqrt{a^2 + x^2 + 2adx} - \sqrt{b^2 + x^2 - 2bdx} < c$ admit the solutions

$$x \in \left(-\infty; \frac{2abd}{a + b}\right) \cup \left(\frac{2abd}{a + b}; +\infty\right).$$

2. $\sqrt{a^2 + x^2 + 2adx} - -\sqrt{b^2 + x^2 - 2bdx} > c$ does not admit any solution.
3. $\sqrt{a^2 + x^2 + 2adx} - -\sqrt{b^2 + x_2 - 2bdx} \leq c$ admit the solutions $x \in R$.
4. $\sqrt{a^2 + x^2 + 2adx} - -\sqrt{b^2 + x_2 - 2bdx} \geq c$ admit the solution $x = \frac{2abd}{a+b}$.
5. Theorem 2 and its corollaries are valid for $a \neq b$. In the case when $a = b$ the triangle ACB becomes isosceles (fig. 5) and the outer bisector from the top of the isosceles triangle becomes parallel to its base. The outer bisector in this case is no longer a segment, but a half-straight, and therefore we cannot determine its length. From this it follows that any isosceles triangle describes an irrational equation, which has no solutions.

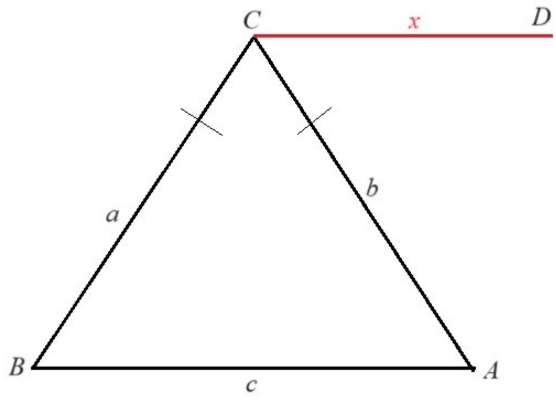

Fig. 5

VI. Problems solved

Problem 1. Solve the following irrational equation:

$$\sqrt{9 + x^2 - 3\sqrt{2}x} + \sqrt{16 + x^2 - 4\sqrt{2}x} = 5.$$

Solution. We construct the right triangle ABC with legs $AC = 4$, $CB = 3$. Then hypotenuse $AB = 5$. Then we take the interior bisector of the right angle $CD = x$ (fig. 6).

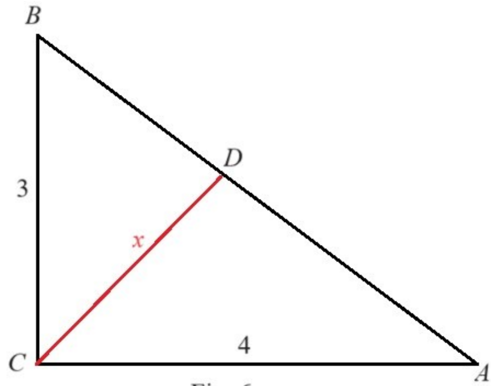

Fig. 6

Solving the irrational equation geometrically means finding the length of the interior bisector CD. We can indicate the following methods.

Method 1. Applying the relationship

$$CD = \frac{AC \cdot CB\sqrt{2}}{AC + CB},$$

we will get $x = \frac{2 \cdot 3 \cdot 4 \cdot \cos 45^0}{3+4} = \frac{12\sqrt{2}}{7}$.

Method 2. Applying the relationship $\frac{BC}{CA} = \frac{BD}{DA}$, we will get $\frac{3}{4} = \frac{\sqrt{9+x^2-3\sqrt{2}x}}{\sqrt{16+x^2-4\sqrt{2}x}}$, from which the equation results $7x^2 - 12\sqrt{2}x = 0$. This equation admits the solutions: $x_1 = 0$ and $x_2 = \frac{12\sqrt{2}}{7}$. But only x_2 is a solution of the initial equation.

Method 3. Note $BD = 3y$ and $DA = 4y$. Then $3y + 4y = 5$ or $y = \frac{5}{7}$. Thus

$$BD = \sqrt{9 + x^2 - 3\sqrt{2}x} = \frac{15}{7}.$$

We obtain the following irrational equation: $x^2 - 3\sqrt{2}x + \frac{216}{49} = 0$. The solutions of the given equation are: $x_1 = \frac{9\sqrt{2}}{7}$ and $x_2 = \frac{12\sqrt{2}}{7}$.

We can convince ourselves that only $\frac{12\sqrt{2}}{7}$ is the solution of the initial irrational equation.

Method 4. Applying the relationship $BD = \sqrt{AB \cdot BC - AD \cdot BD}$, we will get the solution $x = \frac{12\sqrt{2}}{7}$.

Method 5. According to the theorem of sines, from the triangle *CDB* we obtain the following equation: $\frac{x}{\frac{4}{5}} = \frac{\sqrt{9+x^2-3\sqrt{2}x}}{\frac{\sqrt{2}}{2}}$. The solution is $x = \frac{12\sqrt{2}}{7}$.

Problem 2. Solve the following irrational equation:

$$\sqrt{x^2 + 2x + 4} - \sqrt{x^2 - x + 1} = \sqrt{3}.$$

Solution. We write the above irrational equation in the form:

$$\sqrt{4 + x^2 + 2x \cdot \left(2 \cdot \frac{1}{2}\right)} - \sqrt{1 + x^2 - x \cdot \left(2 \cdot \frac{1}{2}\right)} = \sqrt{3}$$

or

$$\sqrt{4 + x^2 + 4x \cdot \sin 30^0} - \sqrt{1 + x^2 - 2x \cdot \sin 30^0} = \sqrt{3}.$$

We notice that 1, $\sqrt{3}$ and 4 are the sides of a triangle (fig. 7).

We construct the exterior bisector *CD* and denote its length by *x*. The solution of the irrational equation will be the length of the bisector *CD*. Regarding the calculation of the length of the outer bisector, that is, to find the solution of the irrational equation, we can indicate the following methods.

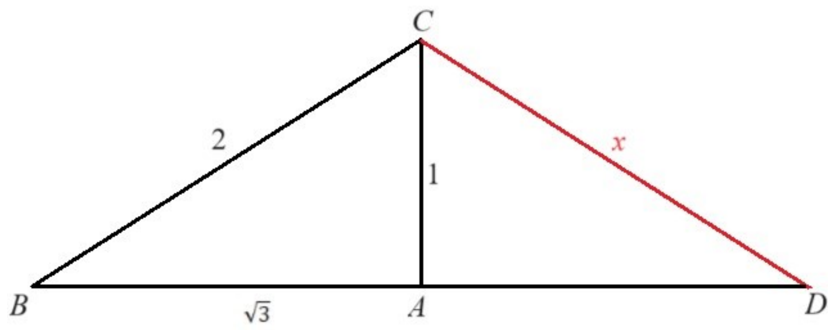

Fig. 7

Method 1. According to the theorem of cosines in the triangle *ACD* we get

$$AD = \sqrt{1 + x^2 - 2x \cdot \sin 3\,0^0} = \sqrt{1 + x^2 - x}.$$

On the other hand

$$AD = \frac{AC \cdot AB}{BC - AC} = \frac{1 \cdot \sqrt{3}}{2 - 1} = \sqrt{3}.$$

Thus $\sqrt{1+x^2-x}=\sqrt{3}$ or $x^2-x-2=0$, where from $x_1=-2$ and $x_2=2$. The solution of the irrational equation is only $x=2$.

Method 2. From the analogous triangle *BCD* we calculate *BD*, ie $BD=\sqrt{x^2+2x+4}$.

But $BD=\frac{BC\cdot AB}{BC-AC}=\frac{2\cdot\sqrt{3}}{2-1}=2\sqrt{3}$. But then the help of the irrational equation

$$\sqrt{x^2+2x+4}=2\sqrt{3}$$

we obtain analogously that $x=2$ is the only solution of the initial equation.

Method 3. We can apply the formula

$$CD=\frac{\sqrt{BC\cdot AC\cdot(AB-AC+BC)\cdot(AB+AC-BC)}}{BC-AC}=$$

$$=\frac{\sqrt{2\cdot1\cdot\left(\sqrt{3}-1+2\right)\cdot\left(\sqrt{3}+1-2\right)}}{2-1}=$$

$$=\sqrt{2\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}=\sqrt{2\cdot2}=2.$$

Method 4. Since *CD* is the exterior bisector of triangle *ABC*, then $\frac{AC}{BC}=\frac{AD}{BD}$. Thus

$$\frac{1}{2}=\frac{\sqrt{x^2-x+1}}{\sqrt{x^2+2x+4}},$$

where from $\sqrt{x^2+2x+4}=2\sqrt{x^2-x+1}$.

Solving the last equation, we will obtain the solution $x=2$.

Method 5. Applying the relation from planimetry, the solution of the initial equation can be calculated according to the formula:

$$x=\frac{2\cdot BC\cdot AC\cdot\sin\frac{60^0}{2}}{BC-AC}=\frac{2\cdot2\cdot1\cdot\frac{1}{2}}{2-1}=2.$$

Method 6. It is easy to find out that $m(\angle ABC)=30^0$. Above we found out that $AD=\sqrt{3}$. Applying the theorem of cosines in the triangle *CBD*, we will get

$$x^2=4+12-8\sqrt{3}\cdot\frac{\sqrt{3}}{2}=16-12=4,$$

where from $x=2$.

Method 7. We apply the theorem of sines in the triangle *CBD*. We will get it

$$\frac{x}{\sin(\angle ABC)}=\frac{BD}{\sin(\angle BCD)}$$

or $\frac{x}{\sin 30^0}=\frac{2\sqrt{3}}{\sin 120^2}$, where from $x=2$.

Method 8. It can be shown that the triangle *CAD* is a right triangle in *A*, and the measure of the angle *ADC* is equal to 30^0. The length of the leg *CA* opposite the angle of 30^0 in a right triangle is twice the length of the hypotenuse *CD*. Therefore $x=CD=2CD=2$.

Method 9. In a triangle *BCD*, the side of the triangle *AC* is the bisector and height, because $\angle BCA \equiv \angle ACD$ and $BA = AD$. This is possible only if the triangle *BCD* is isosceles with the base *BD* and the sides *BC*, *CD*. Therefore $x = CD = BC = 2$.

Method 10. We can apply Stewart's Theorem to the triangle *BCD*, ie

$$BC^2 \cdot AD + CD^2 \cdot BA - CA^2 \cdot BD = BD \cdot BA \cdot AD,$$

$$4\sqrt{3} + \sqrt{3}x^2 - 1 \cdot 2\sqrt{3} = 2\sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3},$$

where from $x^2 = 4$ or $x_1 = -2$ and $x_2 = 2$.

Problem 3. Determine, if possible, the couple $(x; y)$, so that

$$\sqrt{1 + x^2 - \sqrt{2}x} + \sqrt{x^2 + y^2 - xy\sqrt{2}} + \sqrt{2 + y^2 - 2y} = \sqrt{5}.$$

Solution. It can be proved that there is a triangle with sides

$$AB = \sqrt{2},\ BC = 1,\ AC = \sqrt{5} \text{ and } m(\angle ABC) = 135^0.$$

We construct the segments [*BD*] and [*BE*] so that $\angle ABE \equiv \angle EBD \equiv \angle DBC$ and note $BD = x$, $BE = y$ (fig. 8).

With the help of the cosine theorem it is proved that the points $E, D \in (AC)$. The equalities are obtained: $x = \frac{\sqrt{2}y}{1+y}$ and $y = \frac{2x}{\sqrt{2}+x}$.

Thus, we obtain the couple: $\left(\frac{\sqrt{2}}{3}; \frac{1}{2}\right)$.

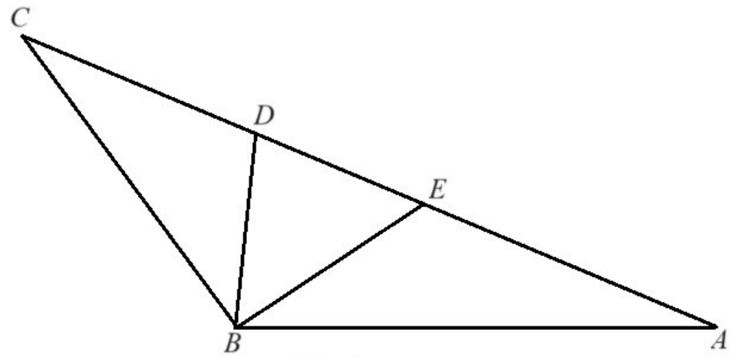

Fig. 8

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Bibliography

1. AFANAS, D.; AFANAS, E. Aplicarea proprietăţilor unei linii importante în triunghi la rezolvarea ecuaţiilor iraţionale. In: *Foaie matematică*. 2001, nr. 5 (2001), pp. 10 – 16. (in romanian)
2. GABOVICI, I. G. *Algoritmiceskii podhod k resheniu geometriceskih zadach*. Kiev: Radianska shkola, 1989, 158 p. (in russian)
3. SHARIGHIN, I. F.; GOLUBEV, V. I. *Fakl'tativnii kurs po matematike*. *Reshenie zadach*. Moskva: Prosveshcenie, 1991, 383p. (in russian)