VARIOUS METHODS OF INTEGRABILITY OF CUBIC DIFFERENTIAL SYSTEMS WITH REAL INVARIANT LINES OF MULTIPLICITY EIGHT

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DIVERSE METODE DE INTEGRABILITATE ALE SISTEMELOR DIFERENȚIALE CUBICE CU DREPTE INVARIANTE REALE DE MULTIPLICITATEA OPT

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Rezumat. În acest articol, se studiază sistemele diferențiale cubice plane cu drepte reale afine invariante de multiplicitate totală opt. Pentru aceste sisteme diferențiale (în total 24 de sisteme diferențiale) este rezolvată problema de integrabilitate prin diverse metode: metoda Darboux, metoda directă și metoda computațională. Rezultatele obținute prin cele trei metode coincid, adică soluțiile obținute sunt echivalente între ele sau diferă între ele printr-o constantă de integrare.
Cuvinte-cheie: Sistem cubic diferențial, dreaptă invariantă, integrabilitate.

Abstract. In this article, we study cubic planar differential systems with affine real invariant straight lines of total multiplicity eight. For these differential systems (there are 24 differential systems) the integrability problem is solved by various methods: the Darboux method, the direct method and the computational method. The results obtained by these three methods coincide, i.e. the obtained solutions are equivalent to each other or they differ from each other by an integration constant.
Keywords: cubic differential system, invariant straight line, integrability.

1. Introduction

Consider the polynomial system of differential equations

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where the functions $P(x, y)$ and $Q(x, y)$ are polynomials in the variables x and y of degree n, where $n = \max\{\deg P, \deg Q\}$ or the differential equation

$$Q(x, y)dx - P(x, y)dy = 0. \quad (2)$$

Let X be the vector field associated with the system (1) and defined by the relation

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

Definition 1. A curve $f = 0$, $f \in C[x, y]$ is called an *algebraic invariant curve* for the system (1) if there exists a polynomial $K_f(x, y) \in C[x, y]$ called the *cofactor* of the algebraic invariant curve, such that $\forall (x, y) \in R^2$ the following identity holds

$$X(f) \equiv f(x,y)K_f(x,y) \Leftrightarrow \frac{\partial f(x,y)}{\partial x}\cdot P(x,y) + \frac{\partial f(x,y)}{\partial y}\cdot Q(x,y) \equiv f(x,y)\cdot K_f(x,y) \quad (3)$$

In particular, if $f(x,y) = ax + by + c, a,b,c \in C$, $(a,b) \neq (0,0)$, then $f = 0$ it is called an *invariant straight line* of system (1).

Definition 2. An algebraic curve $f = 0$ of degree k is invariant for the system (1) with algebraic multiplicity m, if m is the largest natural number such that f^m divides $E_d(X)$, where

$$E_f(X) = \det\begin{pmatrix} v_1 & v_2 & \dots & v_l \\ X(v_1) & X(v_2) & \dots & X(v_l) \\ \dots & \dots & \dots & \dots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \dots & X^{l-1}(v_l) \end{pmatrix}, \quad (4)$$

and $v_1, v_2, ..., v_l$ is a basis for the vector space of polynomials of degree $d : C_d[x,y]$

In the case of invariant straight lines, i.e. $d = 1$, we can take $v_1 = 1$, $v_2 = x$, $v_3 = y$ and

$$E_1(X) = P\cdot X(Q) - Q\cdot X(P), \quad (5)$$

In 1878, Darboux published the paper [2] in which he indicated a general method for integrating differential equations of the form (2) using the algebraic invariant curves of the polynomial systems.

Let the polynomial system (1) have N invariant algebraic curves $f_j(x,y) = 0$, $j = 1,\ldots,N$, i.e. there are cofactors $K_j(x,y)$ $j = 1,\ldots,N$, such that the identities hold:

$$\frac{\partial f_j(x,y)}{\partial x}\cdot P(x,y) + \frac{\partial f_j(x,y)}{\partial y}\cdot Q(x,y) \equiv f_j(x,y)\cdot K_j(x,y) \cdot \quad (6)$$

The idea of Darboux is to search for a first integral of system (1) in the form

$$F(x,y) = \prod_{j=1}^{N} f_j^{\alpha_j}(x,y) = C, \quad (7)$$

where $\alpha_j \in C$ and $f_j(x,y) \in C[x,y]$ A first integral (7) is called a *Darboux first integral*.

Theorem 1. The polynomial system (1) has a Darboux first integral of the form (7) if and only if there exist constants $\alpha_j, j = 1,\ldots,N$, not all identically zero, such that

$$\alpha_1 K_1(x,y) + \alpha_2 K_2(x,y) + \ldots + \alpha_N K_N(x,y) \equiv 0. \quad (8)$$

In case system (1) does not have a first integral of the form (7), Darboux proposes to construct the integrating factor of an analogous form

$$\mu(x,y) = \prod_{j=1}^{N} f_j^{\alpha_j}(x,y), \quad (9)$$

where $\alpha_j \in C$, and $f_j(x,y) = 0$, $j = 1,\ldots,N$ are algebraic invariant curves.

Theorem 2. The polynomial system (1) has a Darboux integrating factor of the form (9) if and only if there exist constants α_j, $j = 1,\ldots,N$, such that

$$\sum_{j=1}^{N} \alpha_j \cdot K_j(x,y) \equiv -\frac{\partial P(x,y)}{\partial x} - \frac{\partial Q(x,y)}{\partial y}. \qquad (10)$$

Theorem 3. If system (1) has N distinct algebraic invariant curves $f_j(x,y)=0$, $j=1,\ldots,N$, where $N \geq \frac{n(n+1)}{2}$, then system (1) has a Darboux first integral of the form (7).

Last years, the Darboux theory of integrability has been developed and extended for invariant algebraic curves taking into account also their multiplicities. Multiple invariant algebraic curves generate exponential functions e^{g_i/h_i}, called *exponential factors*, which are part of the first integral (or integrating factor): $\prod_{j=1}^{N} f_j^{\alpha_j} \prod_{i=1}^{s} e^{g_i/h_i}$. In such situations, one speaks on generalized Darboux integrability.

In [3-8] papers, the classification of cubic differential systems with a number of real and/or complex invariant straight lines by directions has been carried out and the integrability problem has been solved using the Darboux method [2].

2. New results

In this paper we consider the cubic differential system

$$\begin{cases} \dot{x} = p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(x,y), \\ \dot{y} = q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(x,y), \end{cases} \quad GCD(P,Q)=1, \quad (11)$$

where $p_k = \sum_{i+j=k} a_{ij} x^i y^j$ and $q_k = \sum_{i+j=k} b_{ij} x^i y^j$ ($k=\overline{0,3}$) are homogeneous polynomials of degree k in x and y. The coefficients a_{ij} and b_{ij} in polynomials p_k and q_k are assumed to be real, and the condition *GCD(P,Q)*=1 ensures that the right-hand sides of system (11) have not common factors.

Theorem 4. Any cubic differential system with eight real invariant straight lines (including their multiplicities) can be transformed via an affine coordinate transformation and a rescaling of time into one of the 25 systems listed in Table 1 (see [1]).

Tabelul 1. Cubic differential systems with eight real invariant straight lines and their first integrals

No	Differential systems	First integrals/integrating factor
1.	$\begin{cases}\dot{x}=x(x+1)(x-a),\\ \dot{y}=y(y+1)(y-a),\ 0<a\neq 1,\end{cases}$	$\dfrac{(x+1)^{a}(x-a)y^{1+a}}{x^{1+a}(y+1)^{a}(y-a)}=C;$
2.	$\begin{cases}\dot{x}=x^2(x+1),\\ \dot{y}=y^2(y+1),\end{cases}$	$\dfrac{x(y+1)e^{1/x-1/y}}{y(x+1)}=C;$
3.	$\begin{cases}\dot{x}=x(x-1)(x+r),\quad r\neq 0,\\ \dot{y}=y(y-1)[(1-r)x+ry+r],\end{cases}$	$\dfrac{(x-y)(x+ry)^{r}}{y^{1+r}(x-1)(x+r)^{r}}=C;$
4.	$\begin{cases}\dot{x}=rx^3,\quad r\neq 0,\\ \dot{y}=y^2[(r-1)x+y],\end{cases}$	$\dfrac{(x-y)^{r}(rx+y)}{x^{r+1}y^{r+1}}=C;$
5.	$\begin{cases}\dot{x}=(x^2-1)(rx+2y+ry),\\ \dot{y}=(y^2-1)(x+2rx+y),\\ r(r^2-1)(r+2)(2r+1)\neq 0,\end{cases}$	$\dfrac{y+1}{y-1}\cdot\left(\dfrac{x+1}{x-1}\right)^{r}\cdot\left(\dfrac{rx+y-r-1}{rx+y+r+1}\right)^{r+1}=C;$
6.	$\begin{cases}\dot{x}=x^2(rx+2y+ry),\\ \dot{y}=y^2(x+2rx+y),\\ r(r^2-1)(r+2)(2r+1)\neq 0,\end{cases}$	$\dfrac{xy(rx+y)}{(y-x)^2}=C;$
7.	$\begin{cases}\dot{x}=x(x^2-9x-xy-y^2),\\ \dot{y}=-y^2(9+y),\end{cases}$	$\mu=\dfrac{y+9}{x^2y^2(x-y-9)},$ $\dfrac{9}{y}-\dfrac{y+9}{x}+\ln\dfrac{x}{y(x-y-9)}=C;$
8.	$\begin{cases}\dot{x}=x(x^2-xy-y^2),\\ \dot{y}=-y^3,\end{cases}$	$\mu=\dfrac{1}{x^2y(x-y)},$ $\dfrac{y}{x}+\ln\dfrac{(x-y)y}{x}=C;$
9.	$\begin{cases}\dot{x}=x(1-x)(y+1),\\ \dot{y}=y(1-x+y-x^2),\end{cases}$	$\mu=\dfrac{x-y}{x^2y(x-y-1)},$ $\dfrac{y}{x}+x-y+\ln\dfrac{(x-y-1)x}{y}=C;$
10.	$\begin{cases}\dot{x}=(x^2-1)(x+y),\\ \dot{y}=2x(y^2-1),\end{cases}$	$\mu=\dfrac{x-y}{(x^2-1)^2(y^2-1)},$

		$\frac{2(y-x)}{x^2-1}+\ln\frac{(x-1)(y+1)}{(x+1)(y-1)}=C;$
11.	$\begin{cases}\dot{x}=x(x^2-1),\\ \dot{y}=x-y+x^2+3xy,\end{cases}$	$\mu=\frac{x+1}{x^2(x-1)^2},$ $\frac{4x+y+2xy+x^2y}{x(x+1)}-\ln x=C;$
12.	$\begin{cases}\dot{x}=x(x-r+1)(x+r+1),\\ \dot{y}=y(2x-r^2+1),\ \|r\|\neq 1,\end{cases}$	$\left(\frac{y}{x}\right)^{2r}\cdot\frac{(x+r+1)^{1+r}}{(x-r+1)^{1-r}}=C;$
13.	$\begin{cases}\dot{x}=x(1-x+r)(1-x-r),\\ \dot{y}=2y(x+r^2-1),\ \|r\|\neq 1,\end{cases}$	$\frac{x^2y}{(1-x-r)(1-x+r)}=C;$
14.	$\begin{cases}\dot{x}=x(x^2-1),\\ \dot{y}=x-2y,\end{cases}$	$\mu=\frac{1}{x^3},\ \frac{(1-x^2)y}{x^2}-\frac{1}{x}=C;$
15.	$\begin{cases}\dot{x}=x^3,\\ \dot{y}=1+x,\end{cases}$	$y=-\frac{1}{2x^2}-\frac{1}{x}+C;$
16.	$\begin{cases}\dot{x}=x^2(x+1),\\ \dot{y}=1,\end{cases}$	$y+\frac{1}{x}+\ln\frac{x}{x+1}=C;$
17.	$\begin{cases}\dot{x}=x(r+x+x^2),\\ \dot{y}=ry,\ r\leq 1/4,\ r\neq 0,\end{cases}$	$\ln\frac{x}{y\sqrt{x^2+x+r}}-\frac{1}{\sqrt{1-4r}}\ln\left\|\frac{2x+1-\sqrt{1-4r}}{2x+1+\sqrt{1-4r}}\right\|=C,$ $\ln\frac{x}{y(2x+1)}-\frac{1}{2x+1}=C,\ r=\frac{1}{4};$
18.	$\begin{cases}\dot{x}=x(x+1)(x+r),\ r\neq\{0;1\},\\ \dot{y}=y(r+2x+2rx+3x^2),\end{cases}$	$\frac{y}{x(x+1)(x+r)}=C;$
19.	$\begin{cases}\dot{x}=x(1+x)(r+2+(r+1)x),\\ \dot{y}=y(r+2+(3+2r)x+rx^2),\end{cases}$	$\left(\frac{y}{x(x+1)}\right)^{r+1}\cdot(2+r+x+rx)^{r+2}=C;$
20.	$\begin{cases}\dot{x}=x(r+x^2),\\ \dot{y}=1+ry+3x^2y,\ r\leq 0,\end{cases}$	$\frac{2r+3x^2+2r^2y}{x(x^2+r)}+\frac{3}{4r^2\sqrt{-r}}\ln\left\|\frac{x+\sqrt{-r}}{x-\sqrt{-r}}\right\|=C,$ $\frac{y}{x^3}+\frac{1}{5x^5}=C,\ r=0;$

21.	$\begin{cases}\dot{x} = x(1+x),\\ \dot{y} = y(1+x-x^2),\end{cases}$	$\mu = \dfrac{1}{xy(x+1)},\ \dfrac{x(x+1)}{y} = Ce^x;$
22.	$\begin{cases}\dot{x} = x,\\ \dot{y} = -2y-x^3,\end{cases}$	$x^2(5y-x^3) = C;$
23.	$\begin{cases}\dot{x} = x,\\ \dot{y} = y-x^2-x^3,\end{cases}$	$\dfrac{2y}{x}+2x+x^2 = C;$
24.	$\begin{cases}\dot{x} = x(x+1),\\ \dot{y} = y+xy-x^3,\end{cases}$	$\dfrac{y}{x}+x-\ln\|x+1\| = C;$
25.	$\begin{cases}\dot{x} = x^2(x+1),\\ \dot{y} = -1-3x+x^2y-x^3,\end{cases}$	$\dfrac{xy-4x-1}{x(x+1)}+\ln x = C.$

For the class of cubic differential systems with eight real invariant straight lines, the integrability problem has been solved by various methods: the Darboux method (finding the Darboux first integral or the Darboux integrating factor), direct method (reducing the differential system to a first-order differential equation), and computational method (using Wolfram Mathematica software), see Table 1.

3. Justification

We will show various methods of integrability for the differential systems listed in Theorem 4.

Case I. Integrability of System No. 3. Let the differential system be

$$\begin{cases}\dot{x} = x(x-1)(x+r), \quad r \neq 0,\\ \dot{y} = y(y-1)[(1-r)x+ry+r],\end{cases} \tag{12}$$

where $l_1 = x,\ l_2 = x-1,\ l_3 = x+r,\ l_4 = y,\ l_5 = y-1,\ l_6 = x-y,\ l_7 = x+ry$ are the invariant straight lines of this system.

The cofactors of the mentioned straight lines are

$$K_1(x,y) = (x-1)(x+r),\ K_2(x,y) = x(x+r),\ K_3(x,y) = x(x-1),$$
$$K_4(x,y) = (y-1)(x-rx+ry+r),\ K_5(x,y) = y(x-rx+ry+r),$$
$$K_6(x,y) = x^2+xy+ry^2+rx-x-r,\ K_7(x,y) = x^2-rxy+ry^2+rx-x-r$$

respectively. Using the relation (8), we obtain

$$\alpha_1(x-1)(x+r)+\alpha_2 x(x+r)+\alpha_3 x(x-1)+\alpha_4(y-1)(x-rx+ry+r)+$$
$$+\alpha_5 y(x-rx+ry+r)+\alpha_6(x^2+xy+ry^2+rx-x-r)+\alpha_7(x^2-rxy+ry^2+rx-x-r)=0.$$

By equating the coefficients next to the monomials $x^k y^j$, where $k,j=\overline{0,2}$ we will have

$$\begin{array}{l} x^0y^0: \\ x^1y^0: \\ x^0y^1: \\ x^2y^0: \\ x^1y^1: \\ x^0y^2: \end{array} \begin{cases} \alpha_1+\alpha_4+\alpha_6+\alpha_7=0, \\ (r-1)\alpha_1+r\alpha_2-\alpha_3+(r-1)\alpha_4+(r-1)\alpha_6+(r-1)\alpha_7=0, \\ \alpha_5=0, \\ \alpha_1+\alpha_2+\alpha_3+\alpha_6+\alpha_7=0, \\ (1-r)\alpha_4+(1-r)\alpha_5+\alpha_6-r\alpha_7=0, \\ \alpha_4+\alpha_5+\alpha_6+\alpha_7=0. \end{cases} \qquad (13)$$

System (13) is an undetermined compatible system that has the solution

$$\alpha_1=\alpha_5=0,\ \alpha_2=-\alpha_7/r,\ \alpha_3=-\alpha_7,\ \alpha_4=-(r+1)\alpha_7/r,\ \alpha_6=\alpha_7/r.$$

Substituting $\alpha_7=r$ we will have:

$$\alpha_1=\alpha_5=0,\ \alpha_2=-1,\ \alpha_3=-r,\ \alpha_4=-1-r,\ \alpha_6=1,\ \alpha_7=r.$$

Therefore, by substituting the obtained exponents and the invariant straight lines of the system (12) into relation (7), we obtain the Darboux first integral:

$$x^0(x-1)^{-1}(x+r)^{-r}y^{-1-r}(y-1)^0(x-y)^1(x+ry)^r=C \text{ or } \frac{(x-y)(x+ry)^r}{y^{1+r}(x-1)(x+r)^r}=C.$$

Case II. Integrability of System No. 4. Let the differential system be

$$\begin{cases} \dot{x}=rx^3,\ \ r\neq 0, \\ \dot{y}=y^2[(r-1)x+y], \end{cases} \qquad (14)$$

where $l_1=l_2=l_3=x,\ l_4=l_5=y,\ l_6=x-y,\ l_7=\varkappa+y$ are the invariant straight lines of this system.

a) Darboux method. Therefore, the cofactors of the mentioned straight lines are

$$K_1(x,y)=\varkappa^2,\ K_2(x,y)=-\varkappa,\ K_3(x,y)=-2r,\ K_4(x,y)=y(\varkappa-x+y),$$
$$K_5(x,y)=x-\varkappa-y,\ K_6(x,y)=\varkappa^2+rxy+y^2,\ K_7(x,y)=\varkappa^2-y+y^2$$

respectively. Using the relation (8), we obtain

$$r\alpha_1x^2-r\alpha_2-2r\alpha_3+\alpha_4y(\varkappa-x+y)+\alpha_5(x-\varkappa-y)+\alpha_6(\varkappa^2+rxy+y^2)+$$
$$+\alpha_7(\varkappa^2-y+y^2)=0.$$

By equating the coefficients next to the monomials $x^k y^j$, where $k,j=\overline{0,2}$ we will have

$$\begin{array}{l} x^0y^0: \\ x^1y^0: \\ x^0y^1: \\ x^2y^0: \\ x^1y^1: \\ x^0y^2: \end{array} \begin{cases} \alpha_3 = 0, \\ r\alpha_2 + (r-1)\alpha_5 = 0, \\ \alpha_5 = 0, \\ \alpha_1 + \alpha_6 + \alpha_7 = 0, \\ (r-1)\alpha_4 + r\alpha_6 - \alpha_7 = 0, \\ \alpha_4 + \alpha_6 + \alpha_7 = 0. \end{cases} \qquad (15)$$

System (15) is an undetermined compatible system. From system (15), we obtain:

$$\alpha_1 = -(r+1)\alpha_7,\ \alpha_2 = \alpha_3 = \alpha_5 = 0,\ \alpha_4 = -(r+1)\alpha_7,\ \alpha_6 = r\alpha_7.$$

Substituting $\alpha_7 = 1$ we will have: $\alpha_1 = \alpha_4 = -r-1,\ \alpha_2 = \alpha_3 = \alpha_5 = 0,\ \alpha_7 = r.$
Therefore, by substituting the obtained exponents and the invariant straight lines of system (14) into relation (7), we obtain the Darboux first integral:

$$x^{-r-1}e^{0/x}e^{0/x^2}y^{-r-1}e^{0/y}(x-y)^r(rx+y)^1 = C \text{ or } \frac{(x-y)^r(rx+y)}{x^{r+1}y^{r+1}} = C. \quad (16)$$

b) Direct method. The system (15) can be reduced to a differential equation of the form

$$\frac{dy}{dx} = \frac{y^2[(r-1)x+y]}{rx^3} \text{ or } y' = \frac{1}{r}\left(\frac{y}{x}\right)^2 \cdot \left(r-1+\frac{y}{x}\right).$$

The obtained equation is a homogeneous differential equation. We make the substitution $\frac{y}{x} = z,\ y' = z + xz'$. In the result, we obtain

$$z + xz' = \frac{1}{r}z^2(r-1+z) \Rightarrow x\frac{dz}{dx} = \frac{z(z-1)(z+r)}{r} \Rightarrow \frac{dz}{z(z-1)(z+r)} = \frac{dx}{rx}.$$

Integrating both sides, we obtain: $\int \frac{dz}{z(z-1)(z+r)} = \frac{1}{r}\int\frac{dx}{x}.$

We decompose the rational fraction into elementary fractions:

$$\frac{1}{z(z-1)(z+r)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+r} \Rightarrow \frac{1}{z(z-1)(z+r)} = -\frac{1}{rz} + \frac{1}{(r+1)(z-1)} + \frac{1}{r(r+1)(z+r)}.$$

In the result, we obtain:

$$\int\left(-\frac{1}{rz} + \frac{1}{(r+1)(z-1)} + \frac{1}{r(r+1)(z+r)}\right)dz = \frac{1}{r}\int\frac{dx}{x},$$

from which

$$-\frac{1}{r}\text{h}\,|z| + \frac{1}{r+1}\text{h}\,|z-1| + \frac{1}{r(r+1)}\text{h}\,|z+r| = \frac{1}{r}\text{h}\,|x| + \frac{\text{h}\,|\widetilde{C}|}{r},$$

or by performing some elementary transformations, we will have: $\frac{(z-1)^r(z+r)}{z^{r+1}x^{r+1}} = C,$ where $C = \pm\widetilde{C}^{r+1}$. Returning to the substitution $\frac{y}{x} = z$ we obtain $\frac{(x-y)^r(x+y)}{x^{r+1}y^{r+1}} = C.$

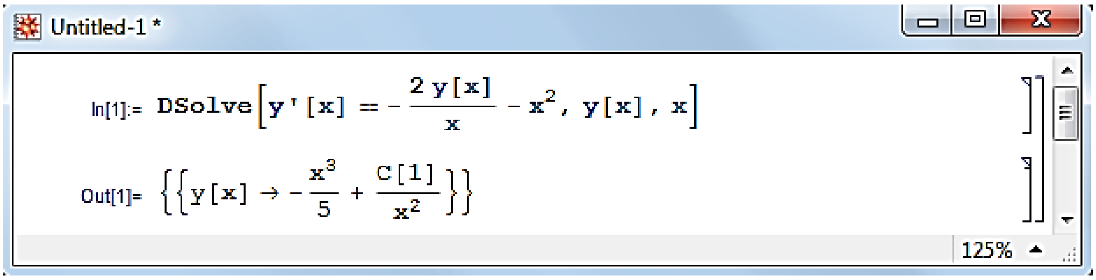
Case III. Integrability of System No. 22. Let the differential system be

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -2y - x^3, \end{cases} \quad (17)$$

which has the invariant straight line $l_1 = x$. System (17) can be brought to the form

$$y' + \frac{2}{x} y = -x^2 \quad (18)$$

being a linear differential equation. Using the Wolfram Mathematica Software, we obtain:



Similarly, using the mentioned methods, the other prime integrals from Table 1 are also demonstrated.

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