QUADRATIC DIFFERENTIAL SYSTEMS WITH THE LINE AT INFINITY OF MAXIMAL MULTIPLICITY

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SISTEME DIFERENȚIALE PĂTRATICE CE AU LINIA DE LA INFINIT DE MULTIPLICITATE MAXIMALĂ

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Rezumat. În lucrarea de față se arată că în clasa sistemelor diferențiale pătratice multiplicitatea maximală a liniei de la infinit este egală cu cinci, iar în cazurile când aceste sisteme au puncte critice cu rădăcinile ecuației caracteristice pur imaginare multiplicitatea liniei de la infinit este egală cu trei și punctele considerate sunt de tip centru.
Cuvinte-cheie: sistem diferențial pătratic, problema centrului, dreaptă invariantă, multiplicitate.

Abstract. In this paper we show that in the class of quadratic differential systems the maximal multiplicity of the line at infinity is five. In the cases when these systems have critical points with purely imaginary eigenvalues the maximal multiplicity of the line at infinity is three and the considered critical points are of the center type.
Keywords: quadratic differential system, center problem, invariant straight line, multiplicity.

1. Introduction

We consider the real polynomial differential system

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \qquad (1)$$

where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$, and $\boldsymbol{\mathcal{X}}$ is the vector field $\boldsymbol{\mathcal{X}} = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\{\deg(p), \deg(q)\}$. If $n = 2$ (respectively, $n = 3, n = 4, n = 5$), then the system (1) is called *quadratic* (respectively, *cubic, quartic, quintic*).

Definition 1. *An algebraic curve $f(x, y) = 0, f \in \boldsymbol{C}[x, y]$, is said to be an invariant algebraic curve of (1) if there exists a polynomial $K_f \in \boldsymbol{C}[x, y]$, such that the identity $\boldsymbol{\mathcal{X}}(f) = f(x, y)K_f(x, y)$ holds.*

In particular, a straight line $L \equiv \alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \boldsymbol{C}$ is called *invariant* for system (1) if there exists a polynomial $K_L \in \boldsymbol{C}[x, y]$ such that the identity holds

$$\alpha p(x, y) + \beta q(x, y) \equiv (\alpha x + \beta y + \gamma)K_L(x, y), (x, y) \in \boldsymbol{R}^2.$$

Definition 2 [1]. *An invariant straight line LL has (algebraic) multiplicity $m(L)$ if $m(L)$ is the greatest positive integer such that $L^{m(L)}$ divides $E(\boldsymbol{\mathcal{X}})$, where*

$$E(\boldsymbol{\mathcal{X}}) = p \cdot \boldsymbol{\mathcal{X}}(q) - q \cdot \boldsymbol{\mathcal{X}}(p).$$

Let $P(x,y,Z), Q(x,y,Z)$ be the homogenized polynomials of $p(x,y), q(xy)$, respectively, and denote $\mathcal{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$.

Definition 3. *We say that the line at infinity $Z=0$ has (algebraic) multiplicity $\nu+1$ if ν is the greatest positive integer such that Z^ν divides $E_\infty(\mathcal{X}_\infty)$, where*

$$E_\infty(\mathcal{X}_\infty) = P\cdot\mathcal{X}_\infty(Q)\, E_\infty(\mathcal{X}_\infty) = P\cdot\mathcal{X}_\infty(Q) - Q\cdot\mathcal{X}_\infty(P).\, Q\cdot\mathcal{X}_\infty(P).$$

Denote by $L_\infty \equiv Z=0$ the line at infinity and by $m(L_\infty)$ the multiplicity of L_∞.

The cubic, quartic and quintic differential systems with the multiple invariant straight lines (including the line at infinity) was investigated in [2-13]. In this paper the quadratic differential systems with the line at infinity are classified and for these systems the problem of the center is solved.

2. The maximal multiplicity of the line at infinity in the class of quadratic systems

Consider the quadratic differential system of the general form

$$\begin{cases}\dot{x} = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 \equiv p(x,y),\\ \dot{y} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 \equiv q(x,y).\end{cases} \quad (2)$$

Suppose that

$$\gcd(p,q) = 1 \text{ and } -b_3x^3 + (a_3-b_4)x^2y + (a_4-b_5)xy^2 + a_5y^3 \not\equiv 0. \quad (3)$$

For system (2) the polynomial $E_\infty(\mathcal{X}_\infty)$ look as

$$E_\infty(\mathcal{X}_\infty) = C_2 + E_\infty(\mathcal{X}_\infty) = C_2 + C_3Z + C_4Z^2 + C_5Z^3 + C_6Z^4 + C_7Z^5,$$

where

$$C_2 = \big((a_4b_3 - a_3b_4)x^2 + (a_5b_3 - a_3b_5)xy + (a_5b_4 - a_4b_5)y^2\big)(-b_3x^3 + (a_3-b_4)x^2y + (a_4-b_5)xy^2 + a_5y^3),$$

$$C_3 = (a_1a_3b_3 - a_3^2b_1 - 2a_4b_1b_3 + a_3b_2b_3 - a_2b_3^2 + a_3b_1b_4 + a_1b_3b_4)x^4 + 3(a_3a_4b_2 - a_2a_4b_3 - a_1a_5b_3 + a_5b_2b_3 + a_5b_1b_4 + a_1a_3b_5 - a_3b_2b_5 - a_1b_4b_5)x^2y^2 + (a_4a_5b_1 - a_4^2b_2 - 2a_3a_5b_2 + 4a_2a_5b_3 + a_2a_4b_4 - 2a_5b_2b_4 - 2a_2a_3b_5 - 2a_1a_4b_5 - 2a_5b_1b_5 + a_4b_2b_5 + a_2b_4b_5 + 2a_1b_5^2)xy^3 + (a_5^2b_1 - a_4a_5b_2 + 2a_2a_5b_4 - a_2a_4b_5 - a_1a_5b_5 - a_5b_2b_5 + a_2b_5^2)y^4,$$

$$C_4 = (-2a_3^2b_0 - a_1a_3b_1 - a_4b_1^2 + a_3b_1b_2 + a_1^2b_3 + 2a_0a_3b_3 - 2a_4b_0b_3 - 2a_2b_1b_3 + a_1b_2b_3 + a_3b_0b_4 + a_1b_1b_4 + a_0b_3b_4)x^3 + (-3a_3a_4b_0 - 2a_5b_1^2 - 3a_1a_3b_2 - a_4b_1b_2 + a_3b_2^2 + 3a_1a_2b_3 + 3a_0a_4b_3 - 4a_5b_0b_3 - a_2b_2b_3 - a_4b_0b_4 - a_2b_1b_4 + 2a_1b_2b_4 + a_0b_4^2 + 2a_3b_0b_5 + 2a_1b_1b_5 + 2a_0b_3b_5)x^2y + (-a_4^2b_0 - 2a_3a_5b_0 + a_2a_4b_1 + a_1a_5b_1 - 2a_2a_3b_2 - 2a_1a_4b_2 - 3a_5b_1b_2 + 2a_2^2b_3 + 4a_0a_5b_3 + a_1a_2b_4 + a_0a_4b_4 - 3a_5b_0b_4 - a_1^2b_5 - 2a_0a_3b_5 + 3a_1b_2b_5 + 3a_0b_4b_5)xy^2 + (-a_4a_5b_0 + 2a_2a_5b_1 - a_2a_4b_2 - a_1a_5b_2 -$$

$$-a_5b_2^2+a_2^2b_4+2a_0a_5b_4-a_1a_2b_5-a_0a_4b_5-2a_5b_0b_5+a_2b_2b_5+2a_0b_5^2)y^3,$$

$$C_5=(-3a_1a_3b_0-2a_4b_0b_1-a_2b_1^2+a_3b_0b_2+a_1b_1b_2+3a_0a_1b_3-2a_2b_0b_3+a_0b_2b_3+a_1b_0b_4+a_0b_1b_4)x^2+(-2a_2a_3b_0-2a_1a_4b_0+a_1a_2b_1+a_0a_4b_1-4a_5b_0b_1-a_1^2b_2-2a_0a_3b_2-a_4b_0b_2-a_2b_1b_2+a_1b_2^2+4a_0a_2b_3+a_0a_1b_4-a_2b_0b_4+2a_0b_2b_4+2a_1b_0b_5+2a_0b_1b_5)xy+(-a_2a_4b_0-a_1a_5b_0+a_2^2b_1+2a_0a_5b_1-a_1a_2b_2-a_0a_4b_2-3a_5b_0b_2+2a_0a_2b_4-a_0a_1b_5+3a_0b_2b_5)y^2,$$

$$C_6=(-a_1^2b_0-2a_0a_3b_0-a_4b_0^2+a_0a_1b_1-2a_2b_0b_1+a_1b_0b_2+a_0b_1b_2+2a_0^2b_3+a_0b_0b_4)+(-a_1a_2b_0-a_0a_4b_0-2a_5b_0^2+2a_0a_2b_1-a_0a_1b_2-a_2b_0b_2+a_0b_2^2+a_0^2b_4+2a_0b_0b_5)y,$$

$$C_7=-a_0a_1b_0-a_2b_0^2+a_0^2b_1+a_0b_0b_2.$$

Takingintoaccount(3)andsolvingthesystemofidentities$\{C_2\equiv 0,\ C_3\equiv 0, C_4\equiv 0, C_5\equiv 0\}$, we obtain the following two solutions

$$a_1=a_2=a_3=a_4=a_5=b_2=b_4=b_5=0,\ a_0b_3\neq 0; \qquad (4)$$

$$a_1=a_3=a_4=b_1=b_2=b_3=b_4=b_5=0,\ a_5b_0\neq 0. \qquad (5)$$

Remark 1. *The transformation of coordinates* $x\to y, y\to x$ *reduces the system* $\{(2),(5)\}$ *to the system* $\{(2),(4)\}$.

Under the conditions (4) the polynomial C_6 yields $C_6=2a_0^2b_3x\neq 0$. After the substitution $x\to a_0x, y\to a_0^2b_3y,\ \frac{b_0}{a_0^2b_3}=a, \frac{b_1}{a_0b_3}=b$, the system $\{(2),(4)\}$ can by written in the form

$$\dot{x}=1,\ \dot{y}=a+bx+x^2.\dot{x}=1,\ \dot{y}=a+bx+x^2. \qquad (6)$$

In this way we prove the following theorem.

Theorem 1. *In the class of quadratic systems* $\{(2),(3)\}$ *the maximal multiplicity of the line at infinity is five. Modulo the affine transformation of coordinates and rescaling the coefficient each quadratic system* $\{(2),(3)\}$ *with the line at infinity of multiplicity five can be written in the form* (6).

3. The problem of the center for quadratic systems with the line at infinity of maximal multiplicity

We consider the quadratic system of the form

$$\dot{x}=y+ax^2+cxy+fy^2\equiv p(x,y),\ \dot{y}=-(x+gx^2+dxy+by^2)\equiv q(x,y). \qquad (7)$$

The critical point $(0,0)$ of system (7) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center*. It is well known that $(0,0)$ is a center for system (7) if the system has an axis of symmetry or an analytical integrating factor of the form $\mu(x,y)$ in a neighborhood of $(0,0)$.

In the case of system (7) the inequalities (3) look as

$$\gcd(p,q)=1,\ gx^3+(a+d)x^2y+(b+c)xy^2+fy^3\not\equiv 0 \quad (8)$$

and $E_\infty(\mathcal{X}_\infty)$ is a polynomial of degree three in Z:

$$E_\infty(\mathcal{X}_\infty)=C_2+C_3Z+C_4Z^2+C_5Z^3,$$

where

$$C_2=((ad-cg)x^2+2(ab-fg)xy+(bc-df)y^2)$$
$$(gx^3+(a+d)x^2y+(b+c)xy^2+fy^3),$$
$$C_3=(a^2+ad-2cg-g^2)x^4+(2ab+ac-cd-2ag-dg-4fg)x^3y$$
$$-3(df+cg)x^2y^2+(2ab+bd-cd-2bf-cf-4fg)xy^3$$
$$+(b^2+bc-2df-f^2)y^4,$$
$$C_4=-((c+2g)x+(d+2f)y)(x^2+y^2),$$
$$C_5=-x^2-y^2.$$

Solving in conditions (8) the system of identities $\{C_2\equiv 0, C_3\equiv 0\}$ we obtain the following three solutions:

$$a=b=d=f=g=0, c\neq 0; \quad (9)$$
$$a=b=c=f=g=0, d\neq 0; \quad (10)$$
$$c=b-\frac{a^2}{b}, d=a-\frac{b^2}{a}, f=-a, g=-b. \quad (11)$$

In each set of conditions (9), (10) and (11) the system (7) obtain the form, respectively

$$\dot{x}=y(1+cx),\ \dot{y}=-x, c\neq 0; \quad (12)$$
$$\dot{x}=y,\ \dot{y}=-x(1+dy), d\neq 0; \quad (13)$$
$$\dot{x}=y+(bx-ay)(ax+by)/b,$$
$$\dot{y}=-(x+(bx-ay)(ax+by)/a). \quad (14)$$

Remark 2. *The transformation* $x\to y, y\to x, d\to c$ *reduces the system* (13) *to the system* (12).

Under the conditions (9) (respectively, (10), (11)) the polynomial C_4 look as $C_4=cx\not\equiv 0$ (respectively, $C_4=dx\not\equiv 0, C_4=-(bx-ay)^2(ax+by)/(ab)\not\equiv 0$).

In this way we prove the following theorem.

Theorem 2. *In the class of quadratic systems* $\{(7),(8)\}$ *the maximal multiplicity of the line at infinity is three. Each quadratic system* $\{(2),(3)\}$ *with the line at infinity of multiplicity three has one of the forms* (12), (13), (14).

Remark 3. *For system* (12) (*respectively,* (13), (14))

- the straight line $cx+1=0$ (*respectively,* $dx+1=0, (a^2+b^2)(ax+by)-ab=0$) *is invariant;*
- $\mu=1/(cx+1)$ (*respectively,* $\mu=1/(dx+1),\ \mu=1/((a^2+b^2)(ax+by)-ab)$) *is an integrating factor;*
- *the critical point* $(0,0)$ *is of center type.*

Theorem 3. *Let for quadratic system (2) the eigenvalues of critical point M_0 are purely imaginary and the line at infinity is of the maximal multiplicity $m(L_\infty) = 3$, Then, this system has: 1) an invariant straight line $L = 0$; 2) an integrating factor of the form $\mu = 1/L$; 3) an axis of symmetry; 4) a center at M_0.*

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