INTEGRABILITY OF LOTKA-VOLTERRA DIFFERENTIAL EQUATION VIA INTEGRATING FACTORS

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INTEGRABILITATEA ECUAȚIEI DIFERENȚIALE LOTKA-VOLTERRA CU FACTORI INTEGRANȚI

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Rezumat. Se consideră ecuația diferențială Lotka-Volterra de forma $y(a_2x+b_2y+c_2)dx = x(a_1x+b_1y+c_1)dy$ cu coeficienții $a_1, b_1, c_1, a_2, b_2, c_2$ reali și variabile x, y reale. Această ecuație, introdusă de către Lotka și Volterra, apare în ecologie la modelarea a două specii în interacțiune. În prezent ecuația are mari aplicații în chimie, matematica aplicată şi într-o mare varietate de subiecte din fizică, cum ar fi fizica laserului, fizica plasmei, rețele neuronale, hidrodinamică şi altele. În această lucrare, în funcţie de coeficienții ecuației, se studiază integrabilitatea ecuației folosind diverse metode, inclusiv determinând factorul integrant de forma $\mu = \mathrm{h}_n^{-1}$, unde $h_n(x,y)$ este un polinom de gradul $n, n \in \{1, 2, 3\}$.

Cuvinte-cheie: ecuația diferențială Lotka-Volterra, factor integrant, integrabilitate.

Abstract. We consider the Lotka-Volterra differential equation $y(a_2x+b_2y+c_2)dx = x(a_1x+b_1y+c_1)dy,$ in which the coefficients $a_1, b_1, c_1, a_2, b_2, c_2$ and variables x, y are assumed to be real. This equation introduced by Lotka and Volterra appears in ecology where it models two species in competition. It has been widely used in chemistry, applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, neural networks, hydrodynamics etc. In this paper, depending on the coefficients of the equation, we study its integrability by using different methods including the existence of integrating factors of the form $\mu = \mathrm{h}_n^{-1}$, where $h_n(x,y)$ is a polynomial of degree $n, n \in \{1, 2, 3\}$.

Keywords: Lotka-Volterra differential equation, integrating factor, integrability.

1. Introduction

Differential equations are of interest to researchers because of the possibility of using them to study a wide variety of problems in the physical, biological and social sciences. Mathematical models of the processes and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable us to make predictions about how the natural process will behave in various circumstances.

We consider the first order differential equations

$$\frac{y}{x} = \frac{y(a_2x+b_2y+c_2)}{x(a_1x+b_1y+c_1)}, \qquad (1)$$

where the coefficients $a_1, b_1, c_1, a_2, b_2, c_2$ and variables x, y are assumed to be real. The equation (1) introduced by Lotka and Volterra in 1920 appears in chemistry and ecology, where it models two species in competition. It has been widely used in applied mathematics and in a large variety of physical topics such as plasma physics, neural networks, laser physics, hydrodynamics etc [1]. The integrability of equation (1) was examined by many mathematicians using different methods: the integrability via polynomial first integrals and polynomial inverse integrating factors was studied in [1], the Darboux integrability by using invariant straight lines and invariant conics was investigated in [2], the Darboux integrability by using invariant straight lines and invariant cubics was carried out in [3].

In this paper we identify the cases when the differential equation (1) is separable equation, homogeneous equation, linear equation, Bernoulli equation, exact equation and study the existence of integrating factors of the form $\mu = h_n^{-1}$, where $h_n(x, y)$ is a polynomial of degree n in $x$and y, where $n \in \{1, 2, 3\}$.

2. Elementary cases of integrability

For arbitrary coefficients $a_1, b_1, c_1, a_2, b_2, c_2$, there is no general method for solving the equation (1) in terms of elementary functions. We will describe several methods, each of which is applicable to a certain subclass of the first order equation (1):

1) If $a_2 = b_1 = 0$, then equation (1) is separable equation. It can be solved by separating variables

$$\frac{dy}{y(b_2 y + c_2)} = \frac{dx}{x(a_1 x + c_1)} \tag{2}$$

and integrating the equation (2).

2) If $a_2 = b_1 = 0$, then equation (1) is also separable equation. It can be solved by separating variables

$$\frac{dy}{y(a_2 x + c_2)} = \frac{dx}{x(b_1 y + c_1)} \tag{3}$$

and integrating the equation (3).

3) If $c_1 = c_2 = 0$, then (1) becomes a homogeneous equation

$$\frac{dy}{dx} = \frac{y(a_2 x + b_2 y)}{x(a_1 x + b_1 y)}. \tag{4}$$

The equation (4) can always be transformed into separable equations by a change of the dependent variable $y = x \cdot v(x), y' = v(x) + xv'(x)$.

4) If $a_1 = c_1 = 0$, then (1) is a first order linear equation

$$\frac{dy}{dx} = \frac{b_2}{b_1 x} y + \frac{a_2 x + c_2}{b_1 x} \tag{5}$$

and can be solved by method of variation of parameters. Another method is to look for general solution of (5) into the form $y = u(x) \cdot v(x)$, where $u(x)$ and $v(x)$ are unknown differentiable functions.

5) If $b_2 = c_2 = 0$, then (1) is a first order linear equation in $x = x(y)$

$$\frac{dx}{dy} = \frac{a_1}{a_2 y} x + \frac{b_1 y + c_1}{a_2 y}. \tag{6}$$

It can be solved by method of variation of parameters or looking for general solution into the form $x = u(y) \cdot v(y)$, where $u(y)$ and $v(y)$ are unknown differentiable functions.

6) If $b_1 = 0$, then (1) is the Bernoulli equation

$$\frac{dy}{dx} = \frac{a_2x + c_2}{x(a_1x + c_1)}y + \frac{b_2}{x(a_1x + c_1)}y^2. \quad (7)$$

In order to solve (7) we first divide the differential equation by y^2. Then we use the substitution $z = y^{-1}$, $z' = -y^{-2}y'$ to convert (7) into a linear equation in terms of $z = z(x)$.

7) If $a_2 = 0$,then (1) is the Bernoulli equation in $x = x(y)$

$$\frac{dx}{dy} = \frac{b_1y + c_1}{y(b_2y + c_2)}x + \frac{a_1}{y(b_2y + c_2)}x^2. \quad (8)$$

Dividing the equation (8) by $x^2$and then using the substitution $z = x^{-1}$, $z' = -x^{-2}x'$ we convert (8) into a differential linear equation in terms of $z = z(y).z = z(y)$.

8) If $a_2 = -2a_1$, $b_1 = -2b_2$, $c_1 = -c_2$, then (1) is an exact differential equation. There exists a function F of two variables x and y having continuous partial derivatives such that the exact differential equation definition is separated as follows

$$F_x(x, y) = y(-2a_1x + b_2y - c_1), \qquad F_y(x, y) = -x(a_1x - 2b_2y + c_1).$$

Therefore, the general solution of the equation is $b_2xy^2 - a_1x^2y - c_1xy = C$, where C is an arbitrary constant.

9) If the differential equation (1) is not exact, it is possible to make it exact by multiplying using a relevant factor $\mu = \mu(x, y) \neq 0$ which is known as integrating factor for the given differential equation [4], [5].

3. The integrating factors of Lotka-Volterra equation

Let us write the differential equation (1) into the form

$$P(x, y)dx + Q(x, y)dy = 0, \quad (9)$$

where $P(x, y) = y(a_2x + b_2y + c_2)$, $Q(x, y) = -x(a_1x + b_1y + c_1)$. We assume that the equation (9) is not exact and $P(x, y)$, are coprime polynomials.

Definition. An integrating factor for equation (9) on some open set U of $\mathbb{R}^2$ is a C^1 function μ defined on U, not identically zero on U such that

$$P\frac{\partial\mu}{\partial y} - Q\frac{\partial\mu}{\partial x} = \mu\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right). \quad (10)$$

It is known [5] that if $P(x, y)$ and $Q(x, y)$ are C^1 functions on U, and $P^2 + Q^2 \neq 0$, then the equation (9) always has an integrating factor on U. In general, the problem of finding an integrating factor for a given differential equation is very difficult. Leonhard Euler introduced the idea of using an integrating factor to solve a differential equation.

The equation (10) for determining integrating factors for Lotka-Volterra equation (1) can be written in the form

$$x(a_1x+b_1y+c_1)\frac{\partial\mu}{\partial x}+y(a_2x+b_2y+c_2)\frac{\partial\mu}{\partial y}+$$
$$+\mu\big((2a_1+a_2)x+(b_1+2b_2)y+c_1+c_2\big)\equiv 0. \qquad (11)$$

In this paper, we investigate the problem of the existence of integrating factors of the form $\mu=h_n^{-1}$, where $h_n(x,y)$ is a polynomial of degree $n, n\in\{1,2,3\}$:

$$x(a_1x+b_1y+c_1)\frac{\partial h_n}{\partial x}+y(a_2x+b_2y+c_2)\frac{\partial h_n}{\partial y}\equiv$$
$$\equiv h_n\big((2a_1+a_2)x+(b_1+2b_2)y+c_1+c_2\big). \qquad (12)$$

1) Let $n=1$ and $h_1=a_{00}+a_{10}x+a_{01}y$, where $(a_{10},a_{01})\neq 0$. In this case identifying the coefficients of the monomials x^iy^j in (12), we obtain a system of six equations for the unknowns a_{00},a_{10},a_{01} and the coefficients of equation (1):

$$\begin{cases}(c_1+c_2)a_{00}=0,\\ (a_1+a_2)a_{10}=0,\\ (b_1+b_2)a_{01}=0,\\ b_2a_{10}+a_1a_{01}=0,\\ (2a_1+a_2)a_{00}+c_2a_{10}=0,\\ (b_1+2b_2)a_{00}+c_1a_{01}=0.\end{cases} \qquad (13)$$

Solving the system (13) in a_{00},a_{10},a_{01}, we prove that the Lotka-Volterra equation (1) has not integrating factors of the form $\mu=h_1^{-1}$ with $P(x,y),Q(x,y)$ coprime polynomials.

2) Let $n=2$ and $h_2=a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2$, where $(a_{20},a_{11},a_{02})\neq 0(a_{20},a_{11},a_{02})\neq 0$. In this case identifying the coefficients of the monomials x^iy^j in (12), we obtain a system of ten equations for the unknowns $a_{00},a_{10},a_{01},a_{20},a_{11},a_{02}$ and the coefficients of equation (1):

$$\begin{cases}(c_1+c_2)a_{00}=0,\\ (2a_1+a_2)a_{00}+c_2a_{10}=0,\\ a_{00}(b_1+2b_2)+c_1a_{01}=0,\\ (c_1-c_2)a_{20}-(a_1+a_2)a_{10}=0,\\ b_2a_{10}+a_1a_{01}=0,\\ (b_1+b_2)a_{01}-(c_2-c_1)a_{02}=0,\\ a_2a_{20}=0,\\ (b_1-2b_2)a_{20}-a_1a_{11}=0,\\ b_2a_{11}+(2a_1-a_2)a_{02}=0,\\ b_1a_{02}=0.\end{cases} \qquad (14)$$

Solving the system (14) in $a_{00},a_{10},a_{01},a_{20},a_{11},a_{02}$, we determine six cases (Table 1) for Lotka-Volterra equation (1) to have an integrating factor of the form $\mu=h_2^{-1}$.

Table 1. The Lotka-Volterra equation with integrating factors

Case	Lotka-Volterra equation	Integrating factor
(i)	$y(b_2y + c_2)dx = x(a_1x + 2b_2y + c_2)dy$	$\mu = \frac{1}{x^2}$
(ii)	$y(a_2x + c_2)dx = x(b_1y + c_1)dy$	$\mu = \frac{1}{xy}$
(iii)	$y(2a_1x + b_2y + c_2)dx = x(a_1x + c_2)dy$	$\mu = \frac{1}{y^2}$
(iv)	$c_1ydx = x(a_1x + b_1y + c_1)dy$	$\mu = \frac{1}{x(a_1x + b_1y)}$
(v)	$y(a_2x + b_2y + c_2)dx = c_2xdy$	$\mu = \frac{1}{y(a_2x + b_2y)}$
(vi)	$y(b_2y + c_2)dx = x(a_1x + c_2)dy$	$\mu = \frac{1}{(a_1x - b_2y)^2}$

and prove the following Theorem:

Theorem 1. The Lotka-Volterra equation (1) has an integrating factor of the form $\mu = h_2^{-1}$ if one of the following conditions is realized:

(i) $a_2 = 0, b_1 = 2b_2, c_1 = c_2, b_1b_2 \neq 0;$

(ii) $a_2 = b_1 = 0,\ a_1 \neq 0;$

(iii) $a_2 = 2a_1,\ b_1 = 0, c_1 = c_2, a_1a_2 \neq 0;$

(iv) $a_2 = b_2 = 0, c_1 = c_2, a_1b_1c_2 \neq 0;$

(v) $a_1 = b_1 = 0, c_1 = c_2, a_2b_2c_1 \neq 0;$

(vi) $a_2 = b_1 = 0, c_1 = c_2, a_1 \neq 0.$

3) Let $n = 3$ and

$h_3 = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^2 + a_{21}xy + a_{12}y^2 + a_{03}y^3$, where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$. In this case identifying the coefficients of the monomials x^iy^j in (12), we obtain a system of fifteen equations for the unknowns $a_{ij}, i + j = 0, 1, 2, 3$ and the coefficients of equation (1):

$$\begin{cases} (c_1+c_2)a_{00}=0, \\ (2a_1+a_2)a_{00}+c_2a_{10}=0, \\ (b_1+2b_2)a_{00}+c_1a_{01}=0, \\ (c_1-c_2)a_{20}-(a_1+a_2)a_{10}=0, \\ a_{10}b_2+a_{01}a_1=0, \\ (c_2-c_1)a_{02}-(b_1+b_2)a_{01}=0, \\ (2c_1-c_2)a_{30}-a_2a_{20}=0, \\ c_1a_{21}+(b_1-2b_2)a_{20}-a_1a_{11}=0, \\ c_2a_{21}-a_{11}b_2-(2a_1-a_2)a_{02}=0, \\ (2c_2-c_1)a_{03}-b_1a_{02}=0, \\ (a_2-a_1)a_{30}=0, \\ (b_2-b_1)a_{30}=0, \\ (a_2-a_1)a_{12}+(b_1-b_2)a_{21}=0, \\ (a_2-a_1)a_{03}=0, \\ (b_2-b_1)a_{03}=0. \end{cases} \quad (15)$$

Solving the system (15) in $a_{ij}, i+j=0,1,2,3$, we determine ten cases (Table 2) for Lotka-Volterra equation (1) to have an integrating factor of the form $\mu = h_3^{-1}$.

Table 2. The Lotka-Volterra equation with integrating factors

Case	Lotka-Volterra equation	Integrating factor
(i)	$y(a_1x+b_2y)dx = x(a_1x+b_1y+c_1)dy$	$\mu = \dfrac{1}{xy^2}$
(ii)	$y(a_2x+b_1y+c_2)dx = x(a_1x+b_1y)dy$	$\mu = \dfrac{1}{x^2y}$
(iii)	$y(b_2y+c_2)dx = x(b_1y+c_1)dy$	$\mu = \dfrac{1}{xy(c_2+b_2y)}$
(iv)	$y(a_2x+c_2)dx = x(a_1x+c_1)dy$	$\mu = \dfrac{1}{xy(a_1x+c_1)}$
(v)	$y(a_1x+b_1y+c_2)dx = x(a_1x+b_1y+c_1)dy$	$\mu = \dfrac{1}{y(a_2x+b_2y)}$
(vi)	$y(a_2x+b_2y+c_2)dx = x(a_1x+c_2)dy$	$\mu = \dfrac{1}{y(c_2+a_1x)(b_2y+(a_2-a_1)x)}$
(vii)	$y(b_2y+c_1)dx = x(a_1x+b_1y+c_1)dy$	$\mu = \dfrac{1}{y(c_1+b_2y)(a_1x+(b_1-b_2)y)}$
(viii)	$y(a_2x+b_2y)dx = x(a_1x+b_1y)dy$	$\mu = \dfrac{1}{xy((a_1-a_2)x+(b_1-b_2)y)}$
(ix)	$y(b_2y+c_2)dx = x(a_1x+b_2y+c_2)dy$	$\mu = \dfrac{1}{x^2(c_2+b_2y)}$
(x)	$y(a_1x+b_2y+c_1)dx = x(a_1x+c_1)dy$	$\mu = \dfrac{1}{y^2(c_1+a_1x)}$

and prove the following Theorem:

Theorem 2. The Lotka-Volterra equation (1) has an integrating factor of the form $\mu = h_3^{-1}$ if one of the following conditions is realized:

(i) $a_1 = a_2, c_2 = 0,\ b_1 \neq b_2;$

(ii) $b_1 = b_2,\ c_1 = 0,\ a_1 \neq a_2;$

(iii) $a_1 = a_2 = 0,\ b_1 \neq b_2;$

(iv) $b_1 = b_2 = 0,\ a_1 \neq a_2;$

(v) $a_1 = a_2, b_1 = b_2,\ c_1 \neq c_2;$

(vi) $b_1 = 0,\ c_1 = c_2, a_1 b_2 (a_1 - a_2)(a_2 - 2a_1) \neq 0;$

(vii) $a_2 = 0,\ \ c_2 = c_1, a_1 b_2 (b_2 - b_1)(b_1 - 2b_2) \neq 0;$

(viii) $c_1 = \ c_2 = 0, (a_1 - a_2)(b_1 - b_2) \neq 0;$

(ix) $a_2 = 0, b_1 = b_2,\ c_1 = c_2,\ \ b_2 \neq 0;$

(x) $a_2 = a_1, b_1 = 0,\ c_2 = c_1,\ a_1 \neq 0.$

The problem of determining the integrating factors for equation (1) of the form $\mu = h_n^{-1}$ with $n > 3, n \in \mathbb{N}$ is a difficult problem with cumbersome calculations.

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