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# COMITANTS OF LIE ALGEBRA OF ROTATION GROUPS FOR TERNARY SYSTEM WITH QUADRATIC NONLINEARITIES

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**Abstarct**. In this article we study ternary system of differential equations that has as projections various mathematical models. This system is studied using Lie algebra of transformations of rotation groups. Using corresponding Lie operators, the comitants and invariants of ternary system of differential equations are investigated.
**Keywords**: ternary differential system, Lie operators, Lie algebras, rotation groups, comitants and invariants.

**Rezumat**. În acest articol se studiază sistemul ternar de ecuații diferențiale care are ca proiecții diverse modele matematice. Acest sistem este studiat utilizând algebra Lie a transformărilor grupului de rotație. Cu ajutorul operatorilor Lie corespunzători sunt cercetați comitanții și invarianții sistemului ternar de ecuații diferențiale.
**Cuvinte cheie**: sistem diferențial ternar, operatori Lie, algebra Lie, grup de rotație, comitanți și invarianți.

We will examine the ternary system of differential equations

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^j x^\beta \qquad (j,\alpha,\beta = 1,2,3), \tag{1}$$

where the tensor $a^j_{\alpha\beta}$ it is symmetrical to the lower indices after which the total convolution takes place and rotation groups

$$\begin{cases} \bar{x}^1 = x^1\cos\varphi_1 + x^2\sin\varphi_1, \\ \bar{x}^2 = -x^1\sin\varphi_1 + x^2\cos\varphi_1, \\ \bar{x}^3 = x^3, \end{cases} \quad \begin{cases} \bar{x}^1 = x^1\cos\varphi_2 + x^3\sin\varphi_2, \\ \bar{x}^2 = x^2, \\ \bar{x}^2 = -x^1\sin\varphi_2 + x^3\cos\varphi_2, \end{cases}$$

$$\begin{cases} \bar{x}^1 = x^1, \\ \bar{x}^2 = x^2\cos\varphi_3 + x^3\sin\varphi_3, \\ \bar{x}^2 = -x^2\sin\varphi_3 + x^3\cos\varphi_3. \end{cases} \tag{2}$$

$$(0 \le \varphi_i < \pi, \ \ i = 1,2,3)$$

The coefficients of system (1) and phase variables $x^1, x^2, x^3$ take values from the fields of real numbers ℝ.

System (1) is a generalization of several mathematical models, which have a great importance for study of different pandemics in society. For example, this system has as projections the mathematical models that govern the dynamics of tuberculosis [1] and a SIV

[2] in society. The SIR mathematical model (susceptible-infected-removed), what is contained in system (1) some specialists [3] have previously used it to examine the spread of covid in society.

We will mention that different problems of system (1) were studied in Chişinău by method of Lie algebras, the theory of invariants and the theory of orbital dimensions according to the modulus of the centro-affine group. $GL(2,\mathbb{R})$ in thesis papers [4,5,6], where important results have been obtained.

An important place lies to rotation groups (2), when the characteristic equation of system (1) have purely imaginary roots. These groups of transformations in each separate case keep the canonical form of system (1) to which the given system can be brought. In this case system (1) possesses extremely complicated geometric properties, but very useful for practice.

Therefore, it is important to study Lie algebra of transformations (2) on system (1), as well as invariants and comitants of this system in relation to the mentioned algebra for use in the research of given system.

**Lemma 1**. *Lie operators of a linear representation of groups* (2) *in the space of phase variables and coefficients of system* (1) *are the following*

$$X_1 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2} + D_1, X_2 = x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3} + D_2, X_1 = x^3\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} + D_3, \qquad (3)$$

*where*

$$D_1 = (a_2^1 + a_1^2)\frac{\partial}{\partial a_1^1} + (-a_1^1 + a_2^2)\frac{\partial}{\partial a_2^1} + a_3^2\frac{\partial}{\partial a_3^1} + (-a_1^1 + a_2^2)\frac{\partial}{\partial a_1^2} - (a_2^1 + a_1^2)\frac{\partial}{\partial a_2^2} - a_3^1\frac{\partial}{\partial a_3^2} +$$
$$+ a_2^3\frac{\partial}{\partial a_1^3} - a_1^3\frac{\partial}{\partial a_2^3} + (2a_{12}^1 + a_{11}^2)\frac{\partial}{\partial a_{11}^1} + (-a_{11}^1 + a_{22}^1 + a_{12}^2)\frac{\partial}{\partial a_{12}^1} + (a_{23}^1 + a_{13}^2)\frac{\partial}{\partial a_{13}^1} + (-2a_{12}^1 +$$
$$+ a_{22}^2)\frac{\partial}{\partial a_{22}^1} + (-a_{13}^1 + a_{23}^2)\frac{\partial}{\partial a_{23}^1} + a_{33}^2\frac{\partial}{\partial a_{33}^1} + (-a_{11}^1 + 2a_{12}^2)\frac{\partial}{\partial a_{11}^2} + (-a_{12}^1 - a_{11}^2 + a_{22}^2)\frac{\partial}{\partial a_{12}^2} +$$
$$+ (-a_{13}^1 + a_{23}^2)\frac{\partial}{\partial a_{13}^2} - (a_{22}^1 + 2a_{12}^2)\frac{\partial}{\partial a_{22}^2} - (a_{23}^1 + a_{13}^2)\frac{\partial}{\partial a_{23}^2} - a_{33}^1\frac{\partial}{\partial a_{33}^2} + 2a_{12}^3\frac{\partial}{\partial a_{11}^3} + (-a_{11}^3 +$$
$$+ a_{22}^3)\frac{\partial}{\partial a_{12}^3} + a_{23}^3\frac{\partial}{\partial a_{13}^3} - 2a_{12}^3\frac{\partial}{\partial a_{22}^3} - a_{13}^3\frac{\partial}{\partial a_{23}^3},$$

$$D_2=(a_3^1+a_1^3)\frac{\partial}{\partial a_1^1}+a_2^3\frac{\partial}{\partial a_2^1}+(-a_1^1+a_3^3)\frac{\partial}{\partial a_3^1}+a_3^2\frac{\partial}{\partial a_1^2}-a_1^2\frac{\partial}{\partial a_3^2}+(-a_1^1+a_3^3)\frac{\partial}{\partial a_1^3}-a_2^1\frac{\partial}{\partial a_2^3}-$$

$$-(a_3^1+a_1^3)\frac{\partial}{\partial a_3^3}+(2a_{13}^1+a_{11}^3)\frac{\partial}{\partial a_{11}^1}+(a_{23}^1+a_{12}^3)\frac{\partial}{\partial a_{12}^1}+(-a_{11}^1+a_{33}^1+a_{13}^3)\frac{\partial}{\partial a_{13}^1}+a_{22}^3\frac{\partial}{\partial a_{22}^1}+$$

$$+(-a_{12}^1+a_{23}^3)\frac{\partial}{\partial a_{23}^1}+(-2a_{13}^1+a_{33}^3)\frac{\partial}{\partial a_{33}^1}+2a_{13}^2\frac{\partial}{\partial a_{11}^2}+a_{23}^2\frac{\partial}{\partial a_{12}^2}+(-a_{11}^2+a_{33}^2)\frac{\partial}{\partial a_{13}^2}-a_{12}^2\frac{\partial}{\partial a_{23}^2}-$$

$$-2a_{13}^2\frac{\partial}{\partial a_{33}^2}+(-a_{11}^1+2a_{13}^3)\frac{\partial}{\partial a_{11}^3}+(-a_{12}^1+a_{23}^3)\frac{\partial}{\partial a_{12}^3}+(-a_{13}^1-a_{11}^3+a_{33}^3)\frac{\partial}{\partial a_{13}^3}-a_{22}^1\frac{\partial}{\partial a_{22}^3}-(a_{23}^1+$$

$$+a_{12}^3)\frac{\partial}{\partial a_{23}^3}-(a_{33}^1+2a_{13}^3)\frac{\partial}{\partial a_{33}^3},$$

$$D_3=a_3^1\frac{\partial}{\partial a_2^1}-a_2^1\frac{\partial}{\partial a_3^1}+a_1^3\frac{\partial}{\partial a_1^2}+(a_3^2+a_2^3)\frac{\partial}{\partial a_2^2}+\frac{\partial}{\partial a_3^2}+(-a_2^2+a_3^3)\frac{\partial}{\partial a_3^2}-a_1^2\frac{\partial}{\partial a_1^3}+(-a_2^2+$$

$$+a_3^3)\frac{\partial}{\partial a_2^3}-(a_3^2+a_2^3)\frac{\partial}{\partial a_3^3}+a_{13}^1\frac{\partial}{\partial a_{12}^1}-a_{12}^1\frac{\partial}{\partial a_{13}^1}+2a_{23}^1\frac{\partial}{\partial a_{22}^1}+(-a_{22}^1+a_{33}^1)\frac{\partial}{\partial a_{23}^1}-2a_{23}^1\frac{\partial}{\partial a_{33}^1}+$$

$$+a_{11}^3\frac{\partial}{\partial a_{11}^2}+(a_{13}^2+a_{12}^3)\frac{\partial}{\partial a_{12}^2}+(-a_{12}^2+a_{13}^3)\frac{\partial}{\partial a_{13}^2}+(2a_{23}^2+a_{22}^3)\frac{\partial}{\partial a_{22}^2}+(-a_{22}^2+a_{33}^2+a_{23}^3)\frac{\partial}{\partial a_{23}^2}+$$

$$+(-2a_{23}^2+a_{33}^3)\frac{\partial}{\partial a_{33}^2}-a_{11}^2\frac{\partial}{\partial a_{11}^3}+(-a_{12}^2+a_{13}^3)\frac{\partial}{\partial a_{12}^3}+(-a_{13}^2+a_{12}^3)\frac{\partial}{\partial a_{13}^3}+(-a_{22}^2+2a_{23}^3)\frac{\partial}{\partial a_{22}^3}+$$

$$+(-a_{23}^2-a_{22}^3+a_{33}^3)\frac{\partial}{\partial a_{23}^3}-(a_{33}^2+2a_{23}^3)\frac{\partial}{\partial a_{33}^3}. \qquad (4)$$

**Theorem 1**. *Lie operators* (3)–(4) *forms a Lie algebra of transformations* $L_3$, *semisimple, with structure equations*

$$[X_1,X_2]=X_3,\ [X_1,X_3]=-X_2,\ [X_2,X_3]=X_1$$
$$([D_1,D_2]=D_3,\ [D_1,D_3]=-D_2,\ [D_2,D_3]=D_1).$$

Because groups (2) are subgroups of centro-affine group $GL(3,\mathbb{R})$ it is obvious that any comitant and invariant of system (1) in relation to this group [4] is respectively comitant and invariant of system (1) in relation to the groups (2).

**Theorem 2**. *In order that polynomial K in the coefficients and phase variables of system* (1) *to be a comitant of this system with respect to Lie algebra of transformations* $L_3$, *it is necessary and sufficient that it satisfies the equations*

$$X_i(K)=0\ \ (i=1,2,3),$$

*where* $X_i$ *are from* (3)–(4), *and in order that polynomial I in the coefficient of system* (1) *to be an invariant of this system with respect to the above mentioned Lie algebra, it is necessary and sufficient that it satisfies the equations*

$$D_i(I)=0\ \ (i=1,2,3),$$

*where* $D_i$ *are from* (4).

Using Theorem 2 the existence of comitants and invariants of system (1) with respect to Lie algebra of transformations $L_3$ of operators from (3)–(4), which are not comitants and invariants of this system with respect to centro-affine group $GL(3,\mathbb{R})$ [4] it was shown.

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